ON THE MODAL CURVE OF NONLINEAR NORMAL MODES

Chol-Hui Pak* and Sun-Jae Park**

(Received Mech 10, 1988)

In a nonlinear holonomic conservative system having two-degree-of-freedom, the modal curves of normal mode vibrations are investigated investimgated by the harmonic balance method. The general procedure to compute the modal curve is suggested. Even if the linearized frequencies of the system are satisfied with the commensurability condition under which the approaches using the perturbation method have the problem of small divisor, the modal curve can be obtained by this method, provided that the fundamental harmonics are dominant when the normal modes are expanded in Fourier series in time domain. As an example, in a system with cubic nonlinearity the modal curves are computed analytically and numerically to compare both results.

Key Words : Nonlinear Normal Mode, Modal Curve, Harmonic Balance Method, Similar Normal Mode, Nonsimilar Normal Mode, Commensurability Condition

1. INTRODUCTION

In this paper, we shall be interested in deriving the procedure to compute the modal curve of normal mode vibrations in nonlinear conservative systems having two degrees of freedom.

In the normal mode of linear systems, the relation between the generalized coordinates x and y is expressed by $y=px$ or $x=qy$ where p and q are constant for all amplitude of vibrations. In nonlinear normal modes, the relation between x and y is not simple ; Both the slope p(or q) and curvature may be varied as the amplitude of mode increases, if the normal mode is nonsimilar(Rosenberg, 1966).

Rand(1974) has utilized a perturbation method to compute the modal curve of nonsimilar normal modes having sufficiently small amplitudes. He has shown that the modal curve may be expressed in the form $y=p_1x+p_2x^3+p_3x^5\cdots p_ix^{2i-1}$, and that the coefficients p_i, $i=2, 3, \cdots$, may be unbounded if the linearized frequencies satisfy some commensurability conditions.

It will be shown here that the modal curve may be expressed in the same form as the previous work, but the coefficients are bounded, regardless of the ratio of linearized frequencies, provided that the fundamental harmonics are dominant when x and y are expanded in Fourier series in time domain. In particular, in the system of cubic nonlinearity the modal curve tends to be a straight line when the amplitude is sufficiently large. Some examples are shown to compare the analytical results with computer solutions.

*Department of Mechanical Engineering, Inha University, Inchon 402~751, Korea
**Department of Mechanical Engineering, Inha Technical Institute, Inchon 402~751, Korea

2. BASIC THEORY

Consider a nonlinear holonomic conservative system having two-degree-of-freedom. Assume that the kinetic energy T may be expressed in the form

$$T=\frac{1}{2}(\dot{x}^2+\dot{y}^2) \tag{1}$$

where dots denote differentiation with respect to time t, and the potential energy $V(x, y)$ satisfies the proerties

(a) $V(x, y)=V(-x, -y)$

(b) ∇V vanishes only at the origin, and the curves ($V(x,y)=$constant) form a continuum of smooth, simple, non-self-intersecting closed curves containing the origin.

Then there is an energy integral

$$T+V(x, y)=h \tag{2}$$

where h is a constant equal to the total energy of a motion and the equations of motion may be written as

$$\begin{aligned}\ddot{x}+V_x&=0\\ \ddot{y}+V_y&=0\end{aligned} \tag{3}$$

where subscripts denote partial derivatives. For a given h, the motions remain in a closed and bounded region

$$\Gamma(h)=\{(x, y) : h-V(x, y)\geq 0\}$$

The normal mode is a periodic motion which passes through the origin and two rest points. The modal curve traced by $x(t)$ and $y(t)$ of normal mode in the xy-plane is called similar if the modal curve is straight, and nonsimilar otherwise.

Due to the property (a) of $V(x, y)$, every trajectory passes through the origin, and τ is the period of a normal mode. Then it is readily verified that for all t

$$x(-t)=-x(t),\quad y(-t)=-y(t) \tag{4}$$

$$x\left(\frac{\tau}{4}-t\right)=x\left(\frac{\tau}{4}+t\right),\quad y\left(\frac{\tau}{4}-t\right)=y\left(\frac{\tau}{4}+t\right) \tag{5}$$

Therefore, the solution of normal modes may be expanded in the Fourier series given by

$$x(t)=\sum_{n=1}^{\infty} A_n \sin(2n-1)\omega t \tag{6a}$$

$$y(t)=\sum_{n=1}^{\infty} B_n \sin(2n-1)\omega t \tag{6b}$$

where ω is the circular frequency of normal mode, $\omega=\frac{2\pi}{\tau}$.

Substitute (6) into the equations (3) of motion and set the coefficients X_n, Y_n of harmonics equal to zero to obtain

$$\begin{aligned} X_n(\omega,\ A_i,\ B_i)=0,\quad n=1,\ 2,\ 3,\ \cdots,\quad i=1,\ 2,\ 3,\ \cdots \\ Y_n(\omega,\ A_i,\ B_i)=0,\quad n=1,\ 2,\ 3,\ \cdots,\quad i=1,\ 2,\ 3,\ \cdots \end{aligned} \tag{7}$$

where the following relation has been utilized

$$\begin{aligned} &\sin^{\alpha}(2l-1)\theta \sin^{\beta}(2m-1)\theta \sin^{\gamma}(2n-1)\theta \\ &=\sum_{i=1}^{M} C_i \sin(2i-1)\theta \end{aligned} \tag{8}$$

for all natural number l, m, n, α, β, γ, and

$$M=\alpha(2l-1)+\beta(2m-1)+\gamma(2n-1).$$

The harmonic balance method is applicable to compute A_n, B_n and ω. Choose the first N terms of (6). Then from (7) $2N$ equations are obtained in terms of $2N+1$ unknowns A_n, B_n, $n=1, 2, 3, \cdots, N$, and ω. Hence A_n and B_n may be solved in terms of ω. By making the use of (8), (6a) may be rewritten in the form

$$x^{2j-1}(t)\approx\sum_{i=1}^{N} d_{ij}\sin(2i-1)\omega t,\quad j=1,\ 2,\ \cdots,\ N \tag{9}$$

where the harmonics of order higher than $2N-1$ are neglected. Now it is claimed that the modal curve may be approximated in the form

$$y=P_1x+P_2x^3+\cdots+P_Nx^{2N-1}. \tag{10}$$

Substitute (9) into (10) and equate the coefficients of harmonics given by (6b) to obtain the linear equations

$$B_i=\sum_{j=1}^{N} d_{ij}P_j,\quad i=1,\ 2,\ \cdots,\ N. \tag{11}$$

Then P_j are readily computed whenever the $(N\times N)$ matrix $D=(d_{ij})$ is nonsingular.

It can be shown that if the order of magnitude of A_1 is greater than A_n, $n=2, 3, \cdots, N$, then the matrix D is nonsingular. In fact, it is found after some calculations that the *jth* column of D contains homogeneous terms of order $2j-1$ in A_1, A_2, $\cdots$, A_N and that every element of D at the main diagonal and above it contains terms of the highest order A_1^{2j-1} but the one below the main diagonal contains that of A_1^{2n-1}, $n<j$. This implies that the determinant of D does not vanish and the assertion (10) is valid.

By taking the number N sufficiently large, the approximate solution so obtained will approach to the exact solution (6), and the modal curve expressed in (10) becomes an infinite series.

3. THE CHARACTERISTICS OF MODAL CURVES

The coefficient P_1 in (10) is the slope of modal curve measured from the x-axis. The other coefficients P_2, P_3, $\cdots$ may represent the curvedness of modal curve. It will be shown that these coefficients are bounded. The curvature of modal curve is written as

$$\chi=\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}=\frac{6P_2x+20P_3x^3+42P_4x^5+\cdots}{[1+(P_1+3P_2x^2+5P_3x^4+\cdots)^2]^{3/2}}. \tag{12}$$

Therefore, if χ is bounded for all x in the closed interval between zero and the x-amplitude of normal mode, then it is clear that coefficient P_2, P_3, $\cdots$ are bounded. The curvature may be rewritten in the form

$$\chi(t)=\frac{V_y\dot{x}-V_x\dot{y}}{v^3} \tag{13}$$

where v is the velocity of normal mode and χ may be expressed as a function of time along a normal mode. Then $\chi(t)$ is a periodic function and continuous except at $t=\frac{\tau}{4}$ where $v=0$. Since every trajectory intercepts the boundary of $\Gamma(h)$ orthogonally, both the numerator and denominator of (13) vanish at $t=\frac{\tau}{4}$, and hence the L'Hospital's rule may be applicable :

$$\lim_{t\to\frac{\tau}{4}}\chi(t)=\frac{\lim_{t\to\frac{\tau}{4}}\frac{d}{dt}(V_y\dot{x}-V_x\dot{y})}{\lim_{t\to\frac{\tau}{4}}\frac{d}{dt}(v^3)}$$

$$=\left.\frac{V_{xy}\left[\left(\frac{dy}{dx}\right)^2-1\right]+\frac{dy}{dx}(V_{xx}-V_{yy})}{3|\nabla V|}\right|_{t=\frac{\tau}{4}}.$$

Therefore, the function $\chi(t)$ is continuous at every t and is bounded.

It will be shown that the coefficients P_2, P_3, $\cdots$ are bounded. Suppose on the contrary that one of coefficient P_j is unbounded. Then the curvature vanishes at every point of modal curve because the denominator of (12) has higher order of magnitude than that of numerator. And the slope of modal curve, as computed by (10), is infinite. This implies that the resulting modal curve represents the $x\equiv 0$ mode, which occurs in a special case $V_x(x,\ y)\equiv 0$, and this mode would not be represented in the form (10). This contradicts the supposition that P_j is unbounded.

It will be shown that every modal curve asymptotically approaches to a straight line as the amplitude of normal mode becomes sufficiently large, if the solutions, represented by Fourier series (6), is dominated by the fundamental harmonics,

$$A_1^2+B_1^2 \gg A_2^2+B_2^2 > A_3^2+B_3^2 > \cdots. \tag{14}$$

The coefficient $P_1, P_2, \cdots$ may be computed by (11). By the condition (12), the determinant Δ of matrix $D=(d_{ij})$ has been demonstrated to be nonsingular. In fact, the highest order term of Δ is found to be $A_1^{N^2}$. The coefficient P_j may be calculated by

$$P_j=\frac{\Delta_j}{\Delta}, \quad j=1, 2, \cdots, N \tag{15}$$

where

$$\Delta=\begin{vmatrix} d_{11} & d_{12} & \cdots & d_{1N} \\ d_{21} & d_{22} & \cdots & d_{2N} \\ d_{31} & d_{32} & \cdots & d_{3N} \\ \vdots & \vdots & \vdots & \vdots \\ d_{N1} & d_{N2} & \cdots & d_{NN} \end{vmatrix},$$

$$\Delta_j=\begin{vmatrix} d_{11} & d_{12} & \cdots & d_{1j-1} & B_1 & d_{1j+1} & \cdots & d_{1N} \\ d_{21} & d_{22} & \cdots & d_{2j-1} & B_2 & d_{2j+1} & \cdots & d_{2N} \\ d_{31} & d_{32} & \cdots & d_{3j-1} & B_3 & d_{3j+1} & \cdots & d_{3N} \\ \cdots\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ d_{N1} & d_{N2} & \cdots & d_{Nj-1} & B_N & d_{Nj+1} & \cdots & d_{NN} \end{vmatrix}.$$

Further calculations show that the element d_{ij} of matrix D at $i=j+1$ has the order of magnitude A_1^{2j-1}, $j=1, 2, 3, \cdots, N$. Then it is readily found that the highest order term of Δ_j is $A_1^{N^2-(2j-1)} \cdot B_j$. Therefore the order of P_j is $A_1^{-(2j-1)} \cdot B_j$, $j=1, 2, \cdots, N$. This implies that P_j, $j=2, 3, \cdots, N$, vanishes when the amplitude of normal mode is sufficiently large, and hence the modal curve approaches to a straight line.

4. SYSTEMS HAVING CUBIC NON-LINEARITY

Consider a system whose potential energy $V(x, y)$ is the sum of quadratic and homogeneous form of order four in the generalized coordinates x and y written in the form

$$V(x, y)=\frac{1}{2}(\omega_1^2 x^2+\omega_2^2 y^2) + ax^4+bx^3y+cx^2y^2+dxy^3+ey^4 \tag{16}$$

where ω_1 and ω_2 are the linearized natural frequencies. Assume that parameters a, b, c, d and e are such that $V(x, y)$ is positive definite in the whole xy-plane. Then it satisfies the properties (a) and (b), and hence the theories described in sections 2 and 3 are applicable. The equations of motion are written as

$$\begin{aligned} \ddot{x}+\omega_1^2 x+4ax^3+3bx^2y+2cxy^2+dy^3=0 \\ \ddot{y}+\omega_2^2 y+bx^3+2cx^2y+3dxy^2+4ey^3=0 \end{aligned} \tag{17}$$

First, we shall find all the possible combinations of system parameters so that a similar normal mode exists. To compute them we have to make use of the fact that a similar normal mode exists if there is a constant number p or q such that $y=px$ or $x=qy$ satisfies the equation (17) of motion. When $y=px$ is substituted into (17) and $\ddot{x}$ is eliminated, we obtain for all x,

$$(\omega_2^2-\omega_1^2)Px+[M(P)-L(P)P]x^3=0 \tag{18}$$

where

$$\begin{aligned} L(P)&=4a+3bP+2cP^2+dP^3 \\ M(P)&=b+2cP+3dP^2+4eP^3. \end{aligned}$$

Since (18) is a finite power series, each coefficient must vanish to obtain

$$(\omega_2^2-\omega_1^2)P=0 \tag{19a}$$

and

$$M(P)-PL(P)=0. \tag{19b}$$

Consider two cases: $\omega_1 \neq \omega_2$ and $\omega_1=\omega_2$. In the case of $\omega_1 \neq \omega_2$, we have $P=0$ by (19a), and $b=0$ by (19b). This is the $y\equiv 0$ mode. Similarly, by taking $x=qy$, we have the $x\equiv 0$ mode with $d=0$. In the case of $\omega_1=\omega_2$, (19a) is identically fulfilled, and similar modes are obtained by solving the fourth order equation (19b) for P. This equation is identical with that of the associated nonlinear homogeneous system(Rosenberg, 1966) given by

$$\begin{aligned} T&=\frac{1}{2}(\dot{x}^2+\dot{y}^2) \\ V&=ax^4+bx^3y+cx^2y^2+dxy^3+ey^4. \end{aligned} \tag{20}$$

It had been shown elsewhere(Pak and Shin, 1985; Pak and Yun, 1985) that there are at least two distinct real roots in (19b). Then there are generically two or four similar modes, and the non-generical case corresponds to three similar modes containing two simple roots and a double root.

In the case of $\omega_1=\omega_2$, it is not possible to find the similar mode $y\equiv 0$ if $b\neq 0$. In expressing the equation (17) of motion with $b\neq 0$, the selection of coordinates seems to be inadequate if one expects that at the small energy h the normal mode may be described in the neighborhood of $y=0$. However, it is always possible to find an orthogonal transformation of coordinates such that in new coordinates the kinetic energy is written in the form (1), and the potential energy in the form (16) with $b=0$. Let the transformation of coordinates be given by

$$\begin{aligned} x&=\cos\theta\ \bar{x}-\sin\theta\ \bar{y} \\ x&=\sin\theta\ \bar{x}+\cos\theta\ \bar{y} \end{aligned} \tag{21}$$

Substitute (21) into the potential energy $V(x, y)$ of (16) to have the resulting expression in the form

$$\begin{aligned} V(\bar{x}, \bar{y})&=\frac{1}{2}\omega^2(\bar{x}^2+\bar{y}^2) \\ &=\bar{a}\bar{x}^4+\bar{b}\bar{x}^3\bar{y}+\bar{c}\bar{x}^2\bar{y}^2+\bar{d}\bar{x}\bar{y}^3+\bar{e}\bar{y}^4. \end{aligned}$$

Then by taking $\bar{b}=0$, (19b) is obtained where $P=\tan\theta$.

In computing the modal curve of nonsimilar normal modes which is the extension of linearized mode $y\equiv 0$, it is assumed that $\omega_1\neq\omega_2$ and $b\neq 0$. For the first approximation of modal curve, assume the solution in the form

$$x=A_1\sin\omega t+A_2\sin 3\omega t \tag{22a}$$

$$y = Px + ax^3. \tag{22b}$$

Substitute (22b) into the first equation of (17) to obtain

$$\ddot{x} + \omega_1^2 x + L(P)x^3 + aL'(P)x^5 = 0. \tag{23}$$

Then the approximate solution of (23) may be written as (22a) where

$$\omega^2 = \omega_1^2 + \frac{3}{4}L(P)A_1^2 + \frac{5}{8}aL'(P)A_1^4 \tag{24a}$$

$$A_2 = -\frac{1}{8\omega^2}\left[\frac{1}{4}L(P)A_1^3 + \frac{5}{16}aL'(P)A_1^5\right]. \tag{24b}$$

Substitute (22b) into the second of (17) to obtain

$$P\ddot{x} + 3ax^2\ddot{x} + ax^3 + M(P)x^3 + aM'(P)x^5 = 0. \tag{25}$$

Substitute (22a) into (25) and set the coefficients of sin ωt and sin $3\omega t$ equal to zero to obtain

$$\begin{aligned}
&A_1(\omega_2^2 - \omega^2)P + \Big[3\omega^2\Big(\frac{3}{4}A_1^3 - \frac{27}{4}A_1^2 A_2 \\
&\quad + \frac{179}{2}A_1 A_2^2\Big) - M'(P)D_1\Big] \\
&\quad = -M(P)\Big(\frac{3}{4}A_1^3 - \frac{3}{4}A_1^2 A_2 + \frac{3}{2}A_1 A_2^2\Big), \\
&A_2(\omega_2^2 - 9\omega^2)P + \Big[3\omega^2\Big(-\frac{1}{4}A_1^3 + \frac{27}{2}A_1^2 A_2 \\
&\quad + \frac{483}{4}A_2^3\Big) + M'(P)D_2\Big] \\
&\quad = -M(P)\Big(\frac{3}{2}A_1^2 A_2 + \frac{3}{4}A_2^3\Big)
\end{aligned} \tag{26}$$

where

$$D_1 = \frac{5}{8}A_1^5 - \frac{5}{16}A_1^4 A_2 + \frac{15}{4}A_1^3 A_2^2 - \frac{15}{8}A_1^2 A_2^3 + \frac{1}{4}A_1 A_2^4$$

$$D_2 = -\frac{5}{16}A_1^5 + \frac{15}{8}A_1^4 A_2 - \frac{5}{4}A_1^3 A_2^2 + \frac{15}{4}A_1^2 A_2^3 + \frac{5}{4}A_1 A_2^4 + \frac{5}{8}A_2^5.$$

By solving (26), P and a may be found. Since (26) are non-linear and coupled, a closed form of solution is not possible.

We shall restrict our task to find the modal curve of nonlinear normal modes having small amplitudes. Since the mode considered here is the extension of linearized mode $y \equiv 0$, P is assumed to be small. Then the following approximations are possible;

$$L(P) = 4a$$
$$M(P) = b$$
$$\omega^2 = \omega_1^2 + 3aA_1^2$$
$$A_2 = \frac{a}{8\omega^2}A_1^3.$$

Therefore, (26) are rewritten as a linear system of equations

$$\begin{aligned}
&(\omega_2^2 - \omega^2)P + \Big(\frac{9}{4}\omega^2 A_1^2\Big)a = -\frac{3}{4}bA_1^2 \\
&(\omega_2^2 - 9\omega^2)P + \Big(6\frac{\omega^4}{a}\Big)a = -\frac{3}{2}bA_1^2.
\end{aligned} \tag{27}$$

Solve (27) to obtain

$$P = \frac{1}{\triangle}\Big(\frac{9}{2}A_1^2 - 6\frac{\omega^2}{a}\Big)\Big(\frac{3}{4}\omega^2 bA_1^2\Big) \tag{28a}$$

$$a = -\frac{1}{\triangle}(7\omega^2 + \omega_2^2)\Big(\frac{3}{4}bA_1^2\Big) \tag{28b}$$

where

$$\triangle = \omega^2\Big\{\frac{6}{a}\omega^2(\omega_2^2 - \omega^2) - \frac{9}{4}(\omega_2^2 - 9\omega^2)A_1^2\Big\}, \tag{29}$$

provided that $\triangle \neq 0$. It is not difficult to show that $\triangle$ does not vanish except the case of $\omega_1 = \omega_2$, where $\triangle$ is computed as

$$\triangle = \frac{27}{4}aA^4(\omega_1^2 + 3aA_1^2).$$

As described previously, if $\omega_1 = \omega_2$, every normal mode is similar, and hence the case of $\omega_1 = \omega_2$ may be eliminated.

On the other hand, when the amplitude is very large, the approximate modal curve can be calculated through the following procedure: The approximate modal curves are assumed in the form of eq. (22a, b). By substituting eq. (22a) into eq. (22b) and neglecting the high order terms of sine function, we can obtain

$$\begin{aligned}
y &= P(A_1 \sin \omega t + A_2 \sin 3\omega t) \\
&\quad + a(A_1 \sin \omega t + A_2 \sin 3\omega t)^3 \\
&\approx \Big[A_1 P + \Big(\frac{3}{4}A_1^3 - \frac{3}{4}A_1^2 A_2 + \frac{3}{2}A_1 A_2^2\Big)a\Big]\sin \omega t \\
&\quad + \Big[A_2 P + \Big(-\frac{1}{4}A_1^3 + \frac{3}{2}A_1^2 A_2 + \frac{3}{4}A_2^3\Big)a\Big]\sin 3\omega t
\end{aligned} \tag{30}$$

Also, the solution y is assumed in the form

$$y = B_1 \sin \omega t + B_2 \sin 3\omega t \tag{31}$$

as two term approximation of eq. (6b). By comparing the coefficient of corresponding harmonics in eq. (30) and eq. (31),

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \begin{bmatrix} A_1 & \Big(\frac{3}{4}A_1^3 - \frac{3}{4}A_1^2 A_2 + \frac{3}{2}A_1 A_2^2\Big) \\ A_2 & \Big(-\frac{1}{4}A_1^3 + \frac{3}{2}A_1^2 A_2 + \frac{3}{4}A_2^3\Big) \end{bmatrix} \begin{Bmatrix} P \\ a \end{Bmatrix}$$

or

$$P = \frac{(-A_1^3 + 6A_1^2 A_2 + 3A_2^3)B_1 - (3A_1^3 - 3A_1^2 A_2 + 6A_1 A_2^2)B_2}{(-A_1^4 + 3A_1^3 A_2 + 3A_1^2 A_2^2 - 3A_1 A_2^3)}$$

$$a = \frac{4(A_1 B_2 - A_2 B_1)}{(-A_1^4 + 3A_1^3 A_2 + 3A_1^2 A_2^2 - 3A_1 A_2^3)}.$$

We can show that the highest order of magnitude of numerator of a is less than that of denominator. Therefore, the value of a vanishes as A_1 approaches to infinity, under the assumption of condition (14). This means that the modal curve approaches to a straight line as the amplitude becomes sufficiently large.

It is readily seen from (28b) that a vanishes as A_1 approaches to zero. Therefore, it may be concluded that the modal curve of nonsimilar normal mode is close to a straight line if the amplitude is sufficiently small or large, regardless of the commensurability of linearized natural frequencies.

5. NUMERICAL EXAMPLES

We consider the cubic nonlinear two-degree-of-freedom system which has the following coefficients of potential energy :

$$a=2\frac{3}{7},\ b=\frac{1}{7},\ c=4\frac{5}{28},\ d=1,\ e=2.$$

In the case, the numerical analysis by using the 4th order Runge-Kutta method is performed to find nonlinear normal modes $x(t)$ and $y(t)$ when the linearized frequencies have the values ;

$$\omega_1=4\ ;\ \omega_2=0.5\omega_1,\ 2\omega_1,\ 3\omega_1.$$

The trajectories of rest point of normal modes are plotted in Fig. 1, 2, 3, respectively. To express the degree of straight of nonlinear normal modes, the shift of modal curves with respect to a staight line is defined as

$$DS=\frac{\triangle_2}{\triangle_1}=\tan(\theta_2-\theta_1)$$

from Fig. 4.

The normal modes $x(t)$ and $y(t)$, so obtained, are expanded as Fourier series by using the Fourier series algorithm to obtain the coeffients of corresponding harmonics of $x(t)$ and $y(t)$ as follows :

$$x(t)=A_1\sin\omega t+A_3\sin 3\omega t+A_5\sin 5\omega t+\cdots$$
$$y(t)=B_1\sin\omega t+B_3\sin 3\omega t+B_5\sin 5\omega t+\cdots.$$

As a result, when the modal curves are expressed as

$$y=P_1x+P_2x^3+\cdots,$$

the coeffients up to the 3rd order polynomial by the procedure

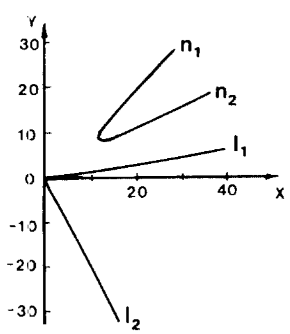

Fig. 1 The trajectories of rest point of normal modes in the case that $a=2\frac{3}{7},\ b=\frac{1}{7},\ c=4\frac{5}{28},\ d=1,\ e=2,\ \omega_1=4,\ \omega_2=0.5\omega_1$

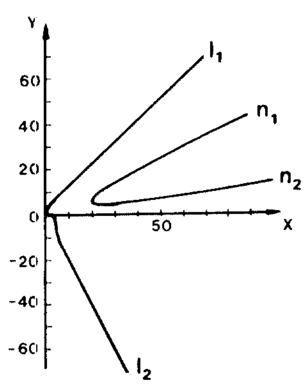

Fig. 2 The trajectories of rest point of normal modes in the case that $a=2\frac{3}{7},\ b=\frac{1}{7},\ c=4\frac{5}{28},\ d=1,\ e=2,\ \omega_1=4,\ \omega_2=2\omega_1$

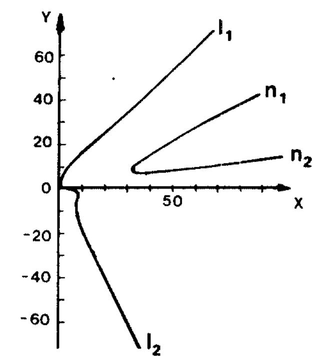

Fig. 3 The trajectories of rest point of normal modes in the case that $a=2\frac{3}{7},\ b=\frac{1}{7},\ c=4\frac{5}{28},\ d=1,\ e=2,\ \omega_1=4,\ \omega_2=3\omega_1$

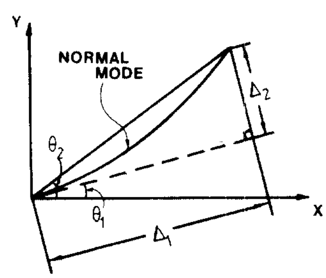

Fig. 4 Nonsimilar normal mode

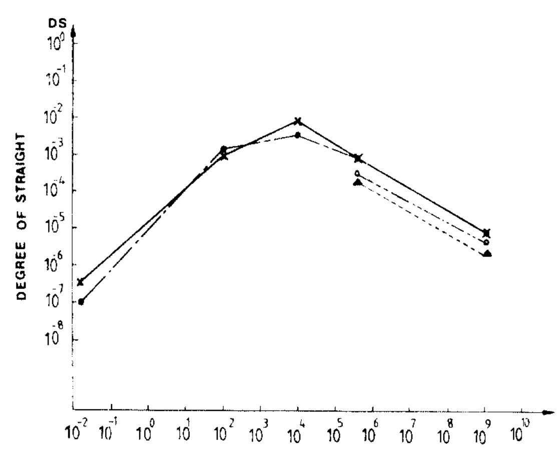

Fig. 5 Degree of straight of modal curves
($a=2\frac{3}{7},\ b=\frac{1}{7},\ c=4\frac{5}{28},\ d=1,\ e=2,\ \omega_1=4,\ \omega_2=2\omega_1$)
● : Mode l_1, extended from a linearized mode
× : Mode l_2, extended from a linearized mode
○ : Bifurcated mode n_1
△ : Bifurcated mode n_2

of section 2 are obtained and tabulated in Table 1, 2, 3. The mode names l_1 and l_2 mean the modes extended from linearized modes, and n_1 and n_2 mean the bifurcated modes in Fig. 1, 2, 3, respectively.

The degrees of straight of modal curves in Table 2 are shown in Fig. 5. It shows that the degrees of straight of modal curves vanish as the amplitudes of normal modes become sufficiently small or large.

Table 1 Fourier coefficients of $x(t)$ and $y(t)$, and the coefficients and the degree of straight of modal curves in the case that $a=2\frac{3}{7}$, $b=\frac{1}{7}$, $c=4\frac{5}{28}$, $d=1$, $e=2$, $\omega_1=4$, $\omega_2=0.5\omega_1$.

h	Fourier coefficients of $x(t)$			Fourier coefficients of $y(t)$			Coefficients of modal curve		DS	Mode name
	A_1	A_3	A_5	B_1	B_3	B_5	P_1	P_2		
0.1E+02	0.97E+0	−0.12E−1	0.29E−3	0.76E−2	−0.25E−3	0.49E−5	0.74E−2	0.66E−3	−6.57E−4	l_1
	0.13E+0	0.11E−2	−0.66E−4	−0.12E+1	0.40E−1	−0.14E−2	−0.85E+1	−0.89E+2	1.67E−2	l_2
0.1E+03	0.21E+1	−0.63E−1	0.26E−2	0.67E−1	−0.26E−2	0.10E−3	0.30E−1	0.23E−3	−1.22E−3	l_1
	0.65E+0	−0.17E−1	0.71E−3	−0.24E+1	0.96E−1	−0.43E−2	−0.35E+1	−0.42E+0	1.38E−2	l_2
0.9E+05	0.13E+2	−0.58E+0	0.25E−1	0.16E+1	−0.74E−1	0.32E−2	0.12E+0	0.98E−6	−1.93E−4	l_1
	0.63E+1	−0.27E+0	0.16E−1	−0.12E+2	0.56E+0	−0.34E−1	−0.20E+1	−0.83E−4	7.52E−4	l_2
	0.10E+2	−0.45E+0	0.22E−1	0.81E+1	−0.36E+0	0.18E−1	0.78E+0	0.98E−5	−7.58E−4	n_1
	0.10E+2	−0.47E+0	0.24E−1	0.75E+1	−0.33E+0	0.17E−1	0.69E+0	0.79E−5	−7.32E−4	n_2
0.1E+10	0.13E+3	−0.55E+1	0.49E+0	0.19E+2	−0.79E+0	0.71E−1	0.14E+0	0.10E−9	−2.05E−6	l_1
	0.66E+2	−0.29E+1	0.15E+0	−0.13E+3	0.58E+1	−0.31E+0	−0.20E+1	−0.71E−8	7.27E−6	l_2
	0.96E+2	−0.39E+1	0.35E+0	0.96E+2	−0.39E+1	0.35E+0	0.99E+0	0.13E−8	−7.45E−6	n_1
	0.12E+3	−0.54E+1	0.24E+0	0.60E+2	−0.27E+1	0.12E+0	0.50E+0	0.43E−9	−5.93E−6	n_2

Table 2 Fourier coefficients of $x(t)$ and $y(t)$, and the coefficients and the degree of straight of modal curves in the case that $a=2\frac{3}{7}$, $b=\frac{1}{7}$, $c=4\frac{5}{28}$, $d=1$, $e=2$, $\omega_1=4$, $\omega_2=2\omega_1$.

h	Fourier coefficients of $x(t)$			Fourier coefficients of $y(t)$			Coefficients of modal curve		DS	Mode name
	A_1	A_3	A_5	B_1	B_3	B_5	P_1	P_2		
0.32E−02	0.15E−7	−0.44E−9	0.28E−11	0.10E−1	0.37E−5	0.20E−5	0.69E+6	−0.27E+21	1.19E−7	l_1
	0.20E−1	0.60E−5	0.34E−5	−0.17E−7	−0.35E−8	−0.57E−11	−0.14E−5	0.17E−2	−6.95E−7	l_2
0.1E+05	0.33E+1	−0.14E+0	0.60E−2	0.66E+1	−0.25E+0	0.10E−1	0.20E+1	−0.27E−2	7.14E−3	l_1
	0.45E+1	−0.19E+0	0.81E−2	−0.65E+1	0.24E+0	−0.98E−2	−0.14E+1	0.12E−2	−1.00E−2	l_2
0.4E+06	0.12E+2	−0.54E+0	0.24E−1	0.14E+2	−0.64E+0	0.29E−1	0.12E+1	−0.20E−4	1.44E−3	l_1
	0.96E+1	−0.42E+0	0.20E−1	−0.18E+2	0.80E+0	−0.37E−1	−0.19E+1	0.63E−4	−1.47E−3	l_2
	0.18E+2	−0.82E+0	0.36E−1	0.52E+1	−0.22E+0	0.10E−1	0.28E+0	−0.21E−5	7.77E−4	n_1
	0.18E+2	−0.82E+0	0.41E−1	0.41E+1	−0.18E+0	0.89E−2	0.22E+0	−0.16E−5	6.27E−4	n_2
0.1E+10	0.96E+2	−0.39E+1	0.35E+0	0.96E+2	−0.39E+1	0.35E+0	0.10E+1	−0.55E−8	2.96E−5	l_1
	0.66E+2	−0.29E+1	0.15E+0	−0.13E+3	0.58E+1	−0.31E+0	−0.19E+1	0.28E−7	−2.90E−5	l_2
	0.12E+3	−0.57E+1	−0.51E+0	0.60E+2	−0.28E+1	−0.25E+0	0.49E+0	−0.16E−8	2.37E−5	n_1
	0.13E+3	−0.64E+1	−0.61E+0	0.19E+2	−0.92E+0	−0.88E−1	0.14E+0	−0.39E−9	8.31E−6	n_2

Table 3 Fourier coefficients of $x(t)$ and $y(t)$, and the coefficients and the degree of straight of modal curves in the case that $a=2\frac{3}{7}$, $b=\frac{1}{7}$, $c=4\frac{5}{28}$, $d=1$, $e=2$, $\omega_1=4$, $\omega_2=3\omega_1$.

h	Fourier coefficients of $x(t)$			Fourier coefficients of $y(t)$			Coefficients of modal curve		DS	Mode name
	A_1	A_3	A_5	B_1	B_3	B_5	P_1	P_2		
0.5E+03	0.84E−1	−0.29E−2	0.63E−4	0.24E+1	−0.18E−1	0.11E−2	0.30E+2	−0.39E+3	4.59E−3	l_1
	0.34E+1	−0.12E+0	0.63E−2	−0.40E−1	−0.89E−3	0.28E−4	−0.13E−1	0.21E−3	−2.85E−3	l_2
0.2E+07	0.18E+2	−0.79E+0	0.39E−1	0.22E+2	−0.96E+0	0.47E−1	0.12E+1	−0.11E−4	1.71E−3	l_1
	0.14E+2	−0.64E+0	0.30E−1	−0.27E+2	0.11E+1	−0.56E−1	−0.19E+1	0.33E−4	−1.76E−3	l_2
0.3E+07	0.20E+2	−0.89E+0	0.47E−1	0.24E+2	−0.10E+1	0.56E−1	0.12E+1	−0.71E−5	1.40E−3	l_1
	0.15E+2	−0.71E+0	0.32E−1	−0.30E+2	0.13E+1	−0.61E−1	−0.19E+1	0.22E−4	−1.43E−3	l_2
	0.30E+2	−0.13E+1	0.73E−1	0.89E+1	−0.38E+0	0.21E−1	0.29E+0	−0.78E−6	7.81E−4	n_1
	0.31E+2	−0.13E+1	0.72E−1	0.66E+1	−0.28E+0	0.15E−1	0.21E+0	−0.55E−6	5.92E−4	n_2
0.1E+11	0.17E+3	−0.81E+1	0.58E+0	0.17E+3	−0.81E+1	0.58E+0	0.10E+1	−0.14E−8	2.49E−5	l_1
	0.11E+3	−0.55E+1	0.27E+0	−0.23E+3	0.11E+2	−0.55E+0	−0.19E+1	0.74E−8	−2.44E−5	l_2
	0.21E+3	−0.10E+2	0.72E+0	0.10E+3	−0.51E+1	0.36E+0	0.49E+0	−0.45E−9	2.00E−5	n_1
	0.23E+3	−0.11E+2	0.81E+0	0.34E+2	−0.16E+1	0.11E+0	0.14E+0	−0.10E−9	7.03E−6	n_2

6. CONCLUSIONS

In this paper, we studied on the modal curves of normal modes of two-degree-of-freedom system. As a result, we obtained the following results:

(1) The method to compute the nonsimilar normal modes is proposed by utilizing the harmonic balance method.

(2) If the fundamental harmonics are dominant when the normal mode $x(t)$ and $y(t)$ are expanded in Fourier series in time domain, the coeffients $P_i(i=2, 3, \cdots)$ of modal curves represented by

$$y = P_1 x + P_2 x^3 + P_3 x^5 + \cdots$$

are bounded regardless of the ratio of linearized frequencies.

(3) The modal curves approach to a straight line as the total energy of the system becomes sufficiently high or low.

(4) The modal curve of a system with the cubic nonlinearity can be approximately considered as a straight line in whole xy-plane.

REFERENCES

Pak, C.H. and Rosenberg, R.M., 1966, "On the Existence of Normal Mode Vibrations in Nonlinear Systems", Quart. Appl. Mech, Vol. 24, pp. 177~193.

Pak, C.H. and Shin, H.J., 1985, "On the Behaviors and Bifurcations of Nonlinear Normal Modes by Harmonic Balance method", Proceedings of the KSME Conference, pp. 313~317.

Pak, C.H. and Yun, Y.S., 1985, "An Approximate Method to Find the Nonlinear Normal Modes by the Principle of Least Action", Proceedings of the KSME Conference, pp. 194~198.

Rand, R.H., 1971, "A Higher Order Approximation for Non-Linear Normal Modes in Two Degrees of Freedom Systems", Int. J. Non-Linear Mechanics, Vol. 6, pp. 545~547.

Rand, R.H., 1974, "A Direct Method for Nonlinear Normal Modes", Int. J. Nonlinear Mech., 9, pp. 363~368.

Rosenberg R.M. and Kuo, J.K., 1964, "Nonsimilar Normal Mode Vibrations of Nonlinear System having Two Degrees of Freedom", J. Appl. Mech. Vol. 31, pp. 283~290.

Rosenberg, R.M., 1966, "On Normal Mode Vibration of System with Many Degrees of Freedom", Academic Press, Appl. Mech. Vol. 9, pp. 155~242.